ON A THEOREM OF V. V. FILIPPOV

BY

DENNIS BURKE AND ERNEST MICHAEL*

ABSTRACT

A new characterization of spaces having a point-countable basis is obtained. This characterization is used in giving a simpler proof of a recent theorem of Filippov.

1. Introduction

The following interesting theorem was recently proved by V. V. Filippov [1, th. 1.1].

THEOREM 1.1. (Filippov). *If X has a point-countable base, and if $f: X \to Y$ is a bi-quotient s-map, then Y has a point-countable base.*[**]

The purpose of this note is to give a new proof of this theorem which is simpler and shorter than Filippov's, and which provides an explicit description of the required base for Y. Our principal tool is a new characterization of spaces with point-countable bases (see Theorem 1.2), which may be of independent interest.

Let us briefly explain our terminology. All maps are continuous and onto. A map $f: X \to Y$ is an *s-map* if every $f^{-1}(y)$ has a countable base. A map $f: X \to Y$ is *bi-quotient* [1] [2] if, whenever $y \in Y$ and $\mathscr{U}$ is a cover of $f^{-1}(y)$ by open subsets of X, then $y \in (\cup f(\mathscr{V}))^\circ$ for some finite $\mathscr{V} \subset \mathscr{U}$. (We use A° to denote the interior of a set A.) No separation properties are assumed.

The class of bi-quotient maps contains all open maps and all perfect maps. It should be remarked that Theorem 1.1 is trivial for open maps (if $\mathscr{B}$ is a point-

* Partly supported by an N.S.F. grant.

** Filippov actually states and proves this result with $\aleph_0$ (as it occurs in the definitions of 'point-countable' and 's-map') replaced by an arbitrary infinite cardinal τ. All results and proofs in this paper are also valid in that generality.

Received March 3, 1971 and in revised form November 30, 1971.

countable base for X, then $f(\mathscr{B})$ is a point-countable base for Y), but for perfect maps the theorem seems no easier to prove than for arbitrary bi-quotient maps.*

Our proof of Theorem 1.1 is based on the following characterization.

THEOREM 1.2. *The following properties of a space Y are equivalent.*

(a) *Y has a point-countable base.*

(b) *Y has a point-countable cover $\mathscr{P}$ such that, if $y \in W$ with W open in Y, then there is a finite subcollection $\mathscr{F}$ of $\mathscr{P}$ such that $y \in (\cup \mathscr{F})^\circ$ and $y \in P \subset W$ for every $P \in \mathscr{F}$.*

In Section 2 we show that Theorem 1.2 implies Theorem 1.1, and in Section 3 we prove Theorem 1.2. Section 4 contains some additional remarks.

2. Proof that Theorem 1.2 implies Theorem 1.1

Let $f: X \to Y$ be a bi-quotient s-map, and $\mathscr{B}$ a point-countable base for X. Define $\mathscr{P} = f(\mathscr{B})$, and let us show that $\mathscr{P}$ satisfies 1.2(b).

Since f is an s-map, each $f^{-1}(y)$ has a countable dense subset, so $f^{-1}(y)$ intersects only countably many elements of $\mathscr{B}$. Hence $\mathscr{P}$ is point-countable.

If $y \in W$ with W open in Y, then the collection $\mathscr{B}'$ of all $B \in \mathscr{B}$ such that $B \subset f^{-1}(W)$ and $B \cap f^{-1}(y) \neq \varnothing$ covers $f^{-1}(y)$. Since f is bi-quotient, $y \in (\cup f(\mathscr{E}))^\circ$ for some finite $\mathscr{E} \subset \mathscr{B}'$, and now $\mathscr{F} = f(\mathscr{E})$ satisfies the requirements of 1.2(b). That completes the proof.

3. Proof of Theorem 1. 2

That 1.2(a) implies 1.2(b) is clear. To prove the converse, assume that Y has a covering $\mathscr{P}$ satisfying 1.2(b), and let us show that Y has a point-countable base.

Let $\Phi = \{\mathscr{F} \subset \mathscr{P}: \mathscr{F} \text{ finite}\}$. For later use, observe that Y must surely be first-countable, for if $y \in Y$, then $\{(\cup \mathscr{F})^\circ: \mathscr{F} \in \Phi,\ y \in \cap \mathscr{F}\}$ is a countable base at y in Y.

To motivate our construction, note that $\{(\cup \mathscr{F})^\circ: \mathscr{F} \in \Phi\}$ is easily seen to be a base for Y, but unfortunately this base need not be point-countable. To obtain a point-countable base, we will shrink the sets $(\cup \mathscr{F})^\circ$ (for $\mathscr{F} \in \Phi$) in a suitable fashion, as follows:

For each $\mathscr{F} \in \Phi$, let

$$\mathscr{M}(\mathscr{F}) = \{A \subset Y: A \subset (\cup \mathscr{F})^\circ,\ A \not\subset (\cup(\mathscr{E})^\circ \text{ if } \mathscr{E} \subsetneq \mathscr{F}\},$$

$$V(\mathscr{F}) = (\cup(\mathscr{M}(\mathscr{F}) \cap \mathscr{P}))^\circ.$$

* However, as observed by Filippov, a perfect map whose domain is T_1 and has a point-countable base is *automatically* an s-map. This follows A. S. Miščenko (4, th. 1).

Let $\mathscr{V} = \{V(\mathscr{F}): \mathscr{F} \in \Phi\}$. We will show that $\mathscr{V}$ is a point-countable base for Y.

LEMMA 3.1. *$\mathscr{V}$ is a base for Y.*

PROOF. Let $y \in W$, with W open in Y. By 1.2(b), there exists an $\mathscr{F} \in \Phi$ such that $y \in (\cup \mathscr{F})^\circ \subset W$, and we may suppose that $y \notin (\cup \mathscr{E})^\circ$ if $\mathscr{E} \subsetneq \mathscr{F}$. Let us show that $y \in V(\mathscr{F}) \subset W$.

Clearly $V(\mathscr{F}) \subset \cup(\mathscr{M}(\mathscr{F})) \subset (\cup \mathscr{F})^\circ \subset W$. To show that $y \in V(\mathscr{F})$, apply 1.2(b) again to pick an $\mathscr{S} \subset \mathscr{P}$ such that $y \in (\cup \mathscr{S})^\circ$ and $y \in P \subset (\cup \mathscr{F})^\circ$ for every $P \in \mathscr{S}$. The latter property of $\mathscr{S}$, together with the fact that $y \notin (\cup \mathscr{E})^\circ$ if $\mathscr{E} \subsetneq \mathscr{F}$, implies that $P \in \mathscr{M}(\mathscr{F})$ or every $P \in \mathscr{S}$. Hence $\mathscr{S} \subset (\mathscr{M}(\mathscr{F}) \cap \mathscr{P})$, so $(\cup \mathscr{S})^\circ \subset V(\mathscr{F})$. Thus $y \in V(\mathscr{F})$, and that completes the proof of the lemma.

It remains to show that $\mathscr{V}$ is point-countable; that is, if $y \in Y$ then $y \in V(\mathscr{F})$ for only countably many $\mathscr{F} \in \Phi$. Now if $y \in V(\mathscr{F})$, then $y \in A$ for some $A \in \mathscr{M}(\mathscr{F}) \cap \mathscr{P}$; since $y \in A$ for only countably many $A \in \mathscr{P}$, it will suffice to prove the following lemma.

LEMMA 3.2. *If $A \subset Y$, then $A \in \mathscr{M}(\mathscr{F})$ for only countably many $\mathscr{F} \in \Phi$.*

PROOF. For each $n \in N$, let $\Phi_n = \{\mathscr{F} \in \Phi: \text{card } \mathscr{F} = n\}$. It clearly suffices to show that $A \in \mathscr{M}(\mathscr{F})$ for only countably many $\mathscr{F} \in \Phi_n$.

Suppose $A \in \mathscr{M}(\mathscr{F})$ for all $\mathscr{F}$ in some uncountable $\Psi \subset \Phi_n$. Pick a maximal $\mathscr{R} \subset \mathscr{P}$ such that $\mathscr{R} \subset \mathscr{F}$ for uncountably many $\mathscr{F} \in \Psi$, and let $\Psi^* = \{\mathscr{F} \in \Psi: \mathscr{R} \subset \mathscr{F}\}$. Clearly $0 \leq \text{card } \mathscr{R} < n$. If $\mathscr{F} \in \Psi^*$, then $A \in \mathscr{M}(\mathscr{F})$ and $\mathscr{R} \not\subset \mathscr{F}$; hence $A \not\subset (\cup \mathscr{R})^\circ$ by definition of $\mathscr{M}(\mathscr{F})$. Pick $y \in A$ such that $y \notin (\cup \mathscr{R})^\circ$. Let $E = Y - \cup \mathscr{R}$, so $y \in \bar{E}$. Since Y is first-countable, $y \in \bar{Z}$ for some countable $Z \subset E$. Now if $\mathscr{F} \in \Psi^*$, then $y \in (\cup \mathscr{F})^\circ$ (since $y \in A \subset (\cup \mathscr{F})^\circ$), so Z intersects some $P \in \mathscr{F}$. But Z intersects only countably many $P \in \mathscr{P}$, and Ψ^* is uncountable, so Z intersects some $P_0 \in \mathscr{P}$ which lies in uncountably many $\mathscr{F} \in \Psi^*$. Note that $P_0 \notin \mathscr{R}$, since P_0 intersects Z while $\cup \mathscr{R}$ does not. Let $\mathscr{S} = \mathscr{R} \cup \{P_0\}$. Then $\mathscr{S} \supsetneq \mathscr{R}$ and $\mathscr{S} \subset \mathscr{F}$ for uncountably many $\mathscr{F} \in \Psi$, which contradicts the maximality of $\mathscr{R}$. That completes the proof of our lemma, and thus also of Theorem 1.2.

4. Further remarks

(4.1). As the proof shows, Lemma 3.2 is valid under the following two assumptions on $\mathscr{P}$ and Y (both of which follow from 1.2(b)).

(a) $\mathscr{P}$ is point-countable.

(b) If $y \in \bar{E}$ in Y, then $y \in \bar{Z}$ for some countable $Z \subset E$.

(4.2) If Y is discrete (thus surely satisfying 4.1(b)), Lemma 3.2 reduces to the following result of A.S. Miščenko [4]: *If $\mathscr{P}$ is a point-countable cover of a set Y then every $A \subset Y$ has only countably many minimal finite covers by elements of $\mathscr{P}$.*

(4.3) Condition 4.1(b) is clearly satisfied by all first-countable spaces and is preserved under quotient maps [3, 8.2–8.5]. Our arguments therefore establish the following generalization of Theorem 1.1. *If X has a point-countable base, and if $f: X \to Y$ is a quotient s-map, then Y has a point-countable open cover which is a base at every $y \in Y$ where f is bi-quotient.*

(4.4) Theorem 1.2 becomes false if 1.2(b) is weakened by not requiring that $y \in P$ for every $P \in \mathscr{F}$. In fact, let Y be set of ordinals $[0, \omega_1]$, topologized by giving ω_1 the usual neighborhoods and making $\{\alpha\}$ open if $\alpha < \omega_1$. Let $\mathscr{P}$ consist of all singletons in Y and all intervals $[\alpha, \omega_1)$ with $\alpha < \omega_1$. Then $\mathscr{P}$ satisfies the above weakned form of 1.2(b), but Y is not even first-countable.

References

1. V. V. Filippov, *Quotient spaces and multiplicity of a base*, Mat. Sb. **80** (1969), 521–532. [Math. USSR-Sb. **9**(1969), 487–496].

2. E. Michael, *Bi-quotient maps and cartesian products of quotient maps*, Ann. Inst. Fourier (Grenoble) **18** (1968), 287–302.

3. E. Michael, *A quintuple quotient quest*, to appear in Gen. Topology.

4. A. S. Miščenko, *Spaces with a pointwise denumerable basis* (Russian), Dokl. Akad. Nauk. SSSR **145** (1962), 985–988 [Soviet Math. Dokl. **3** (1962), 855–858].